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Existence of solutions with low
regularity for some non-linear
systems of differential equations

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This dissertation consists of a summary of the following two papers:

- [A] Källén, A.: Isometric embedding of a smooth compact manifold with a metric of low regularity. Arkiv för Matematik, Vol. 16 (1978), 29-50.
- [B] Källén, A.: A remark on the C^1 isometric embedding theorem of Nash. University of Lund, Department of Mathematics, Preprint 5 (1978).

Let M be a Riemannian manifold, that is, a manifold with a positive definite metric g . To find a concrete realization of M as a submanifold of $\mathbb{R}^N$ one must find an isometric embedding $u \in C^1(M, \mathbb{R}^N)$ of M in $\mathbb{R}^N$, that is, an injective solution of the non-linear system of differential equations

$$(1) \quad \langle du, du \rangle = g.$$

Given the regularity of g one should also examine what regularity one can obtain for u .

Local existence theorems were proved in the analytic case by E. Cartan and Janet in the 1920's. In the middle of the 1950's Nash published two papers on the global existence question when $g \in C^k$. In [7] he showed that if $g \in C^0$ there is an isometric embedding $u \in C^1(M, \mathbb{R}^N)$ provided that $N \geq n+2$ and that there is a smooth embedding of M in $\mathbb{R}^N$, in particular if $N \geq 2n$. Nash also indicated that the condition $N \geq n+2$ could be weakened to $N \geq n+1$, which was proved by Kuiper [4]. It should be observed that (1) in local coordinates means $n(n+1)/2$ equations for N variables. For $g \in C^k$, $k \geq 3$, Nash [8] also showed that there is an isometric embedding $u \in C^k(M, \mathbb{R}^N)$ if $N \geq n(3n+11)/2$. This result of Nash was extended by Jacobowitz [3] to Hölder classes H^a with $a > 2$,

and he also showed that there are metrics $g \in H^\beta$, $\beta > 2$, such that (1) has no solution $u \in H^\alpha(M, \mathbb{R}^N)$, $\alpha > \beta$, for any N . In [A] we consider the embedding problem for $0 < \beta \leq 2$ and prove that if N is sufficiently large and $\alpha < 1 + \beta/2$, there is $u \in H^\alpha(M, \mathbb{R}^N)$ which solves (1). We also prove that if $\alpha > 1 + \beta/2$ there are metrics $g \in H^\beta$, $0 \leq \beta < 2$, such that (1) has no solution $u \in H^\alpha(M, \mathbb{R}^N)$ for any N . We leave it as an open question whether for every $g \in H^\beta$, $0 < \beta \leq 2$, there is an isometric embedding $u \in H^\alpha$, $\alpha = 1 + \beta/2$.

Let us make a few remarks on the proofs of some of the theorems mentioned above. Suppose that $g \in H^\beta$ is given. Then we want to construct a solution $u \in H^\alpha$ of (1) by means of an iteration scheme producing smooth metrics g_k and functions u_k such that $g_k \rightarrow g$, $u = \lim u_k \in H^\alpha$ and

$$(2) \quad \langle du_k, du_k \rangle = g_k + e_k.$$

Here the error term e_k is to be so small that it almost can be corrected in the next step. For $v_{k+1} = u_{k+1} - u_k$, (2) means that we want to find v_{k+1} such that approximately

$$2\langle du_k, dv_{k+1} \rangle + \langle dv_{k+1}, dv_{k+1} \rangle = g_{k+1} - (g_k - e_k).$$

Calling the right hand side h and dropping the index k this means that, to given smooth u and h , we want to find v such that

$$(3) \quad 2\langle du, dv \rangle + \langle dv, dv \rangle = h,$$

at least approximately. Suppose that we try v of the form

$$v(x) = V(x, \theta\phi(x))/\theta.$$

with θ large. Modulo terms of order $O(\theta^{-1})$, (3) then becomes

$$(4) \quad 2\langle du, w(\cdot, \theta\phi) \rangle d\phi + \langle w(\cdot, \theta\phi), w(\cdot, \theta\phi) \rangle (d\phi)^2 = h$$

where $w = \partial_y V$. One way to solve (4) is to find w so that

$$\langle du(x), w(x,y) \rangle = 0, \langle w(x,y), w(x,y) \rangle = 1, x \in M, y \in \mathbb{R}.$$

In addition w should be periodic in y with mean value 0 so that a bounded integral v exists. Nash satisfied these conditions by taking orthonormal normals ξ and η to $u(M)$ and defining

$$w(x,y) = (\sin y)\xi(x) + (\cos y)\eta(x).$$

With this choice of w , (4) becomes $h = (d\phi)^2$, so we can (see below) find v when h is the square of an exact one form. A subdivision of the k^{th} step shows that we can solve (3) for h in the interior of the cone consisting of positive linear combinations of such forms, that is, for h positive definite. If we take for v the periodic integral of w whose mean value is 0, a differentiation gives

$$dv(x) = w(x, \theta\phi(x))d\phi(x) + O(\theta^{-1}).$$

We thus have all the control we want on dv (since $|w| \leq 1$), but none on higher derivatives. We therefore get a solution of (1) which is in C^1 , using only the fact that $g \in C^0$. This is the method of proof in [7] and in [B] we try to generalize these ideas to more general operators. There we consider operators Φ of order m from sections of one vector bundle E over M to another F . We use the principal part of the linear and the quadratic parts of its Taylor expansion to define a conic set in F , corresponding to the cone of positive definite forms in $S^2(T^*M)$ for the embedding operator. If $f \in C^0(F)$ has its image in this set, we then prove that the equation

$$\Phi(u) = \Phi(0) + tf$$

has a solution $u \in C^m$ for some $t > 0$ if M is compact.

Let us now return to the construction of an isometric embedding $u \in H^\alpha$ when $\alpha > 1$ and M is compact. If $u \in C^V$ the normals ξ and η will only belong to C^{V-1} and so will v . So we lose derivatives in the iteration process. Nash [8] overcame this problem by taking normals not to $u(M)$ but to $S_\theta u(M)$, where S_θ is a smoothing operator. Let therefore $\{\xi_s\}$ be an orthonormal set of normals to $S_\theta u(M)$ and let v be of the form

$$v = \sum c_s \xi_s,$$

where c_s are real valued functions on M . If we neglect some small terms, (3) then means that we want to find c_s such that approximately

$$(5) \quad \sum ((dc_s)^2 + 2c_s \langle dS_\theta u, d\xi_s \rangle) = h.$$

How one should attack this problem now depends on the Hölder exponent β of g .

For $\beta < 2$ the dominant term is the quadratic. Write (formally) half the functions c_s as $a_t \cos(\theta \phi_t)/\theta$ and the other half as $a_t \sin(\theta \phi_t)/\theta$. Then

$$\sum (dc_s)^2 = \sum (a_t^2 (d\phi_t)^2 + (da_t/\theta)^2),$$

and since any positive definite form is the sum of n squares of linear forms we can as above solve the system

$$\sum a_t^2 (d\phi_t)^2 = h.$$

With a rather heavy use of the inverse function theorem, we can then solve (5) with the required accuracy. This is the method of proof in [A].

Note that with increasing regularity on g , the linear part in (3) becomes progressively more important. In fact, for $\beta > 2$ one can omit $\langle dv, dv \rangle$ in (3) and, as before, impose the side condition that the perturbation v is to be normal to $u(M)$. Then (3) becomes the linear system

$$\langle d^2u, v \rangle = -h/2, \quad \langle du, v \rangle = 0,$$

whose solution $v = \Psi(u)h$ is a linear operator with coefficients that are second order differential operators in u . But as before, if we use this directly in the iteration process we will lose derivatives. Therefore we define $v = \Psi(S_\theta u)h$ instead in the process. Note that since the linear term is now dominating we do not have to have h positive definite in this situation. Many elaborations of this idea of Nash have been made (for details see Hamilton [1], Hörmander [2], Moser [5,6] and Zehnder [9]).

Finally I want to express my gratitude to Professor Lars Hörmander, who suggested these problems to me and whose advice and encouragement have been invaluable in the progress of the work with this thesis.

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A REMARK ON THE C^1 ISOMETRIC EMBEDDING
THEOREM OF NASH

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A REMARK ON THE C^1 ISOMETRIC EMBEDDING

THEOREM OF NASH

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1. Introduction. Let M be a C^∞ manifold of dimension n with a C^k Riemannian metric g . The isometric embedding problem is then the study of the equation

$$(1.1) \quad \langle du, du \rangle = g,$$

where $u \in C^1(M, \mathbb{R}^N)$ and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^N$. In the middle of the 50's Nash proved some existence theorems for this equation. In [3] he proved that if $g \in C^0$ we can find an injective $u \in C^1(M, \mathbb{R}^N)$ provided that $N \geq n+2$ and that there is a smooth embedding of M in $\mathbb{R}^N$. Two years later, in [4], he proved that if $g \in C^k$, $k > 2$, we can find an injective solution $u \in C^k(M, \mathbb{R}^N)$ of (1.1) if $N \geq n(3n+11)/2$. This latter result, as is well known, used techniques which lent themselves readily to much more general problems. A general formulation of these results can be found in [1] and the references there.

This note is the outcome of an attempt to analyse the method of proof in [3]. The result is formulated in two parts, the definitions 2.1 - 2.3 and Theorem 3.1. Because of the intricacy of these definitions a few general and non-precise preparatory remarks on the problem solved by Nash might not be amiss.

To solve the equation (1.1) one needs a method that, for given u , gives a good approximate solution v to the equation

$$\langle d(u+v), d(u+v) \rangle = \langle du, du \rangle + h, \text{ that is,}$$

$$(1.2) \quad 2\langle du, dv \rangle + \langle dv, dv \rangle = h.$$

Nash had two different approaches to this problem. In [4] he neglected the term $\langle dv, dv \rangle$ and solved the equations

$$2\langle du, dv \rangle = h, \quad \langle du, v \rangle = 0.$$

In [3] Nash had another approach, namely to emphasize the quadratic term $\langle dv, dv \rangle$ by making $\langle du, dv \rangle$ and $\langle dv, dv \rangle - h$ small. If we try to find v of the form

$$v(x) = \theta^{-1} V(x, \theta \phi(x)),$$

where $V(x, y)$ and ϕ are smooth and θ is large, then neglecting terms of magnitude $O(\theta^{-1})$ leads to the equations

$$\langle du, W(\cdot, \theta \phi) \rangle d\phi = 0, \quad \langle W(\cdot, \theta \phi), W(\cdot, \theta \phi) \rangle (d\phi)^2 = h,$$

for $W = \partial_y V$. Thus we want $W(x, y)$ to be normal to $u(M)$ for every y and also, for instance, that $\langle W(x, y), W(x, y) \rangle = 1$ for all x and y . But these conditions are easy to satisfy, using the fact that $\sin^2 + \cos^2 = 1$; taking ζ and η to be orthonormal normals to $u(M)$ we can define as Nash did

$$W(x, y) = (\sin y)\zeta(x) + (\cos y)\eta(x).$$

With this choice, (1.2) becomes approximately $(d\phi)^2 = h$, so the method doesn't solve (1.2) for every h , but only for positive h of rank one. Actually it is clear that a slight extension of the idea shows how to solve (1.2) for every h in the interior of the convex hull of this cone, that is, the positive definite sections h .

Finally I want to thank Professor Hörmander for the invaluable help he has given me in the progress of this work.

2. Definitions. The setup is as follows. Let M be a smooth compact manifold of dimension n and let E and F be two vector bundles on M with fiber dimensions N and N' respectively. If G is a vector bundle over M , let $C^r(G)$ denote the space of r times continuously differentiable sections of G . If $u \in C^k(E)$ we let $j^k u$ denote the k -jet of u and $J^k(M, E)$ the corresponding jet bundle considered as a vector bundle over M . Let $S^k(E^*)$ denote the vector bundle over M whose fibers are the spaces $S^k(E_x^*)$ of symmetric k -linear forms on E_x . Then we have an exact sequence

$$0 \longrightarrow S^k(T^*M) \otimes E \xrightarrow{i} J^k(M, E) \xrightarrow{p} J^{k-1}(M, E) \longrightarrow 0,$$

where $p(j^k u(x)) = j^{k-1} u(x)$ and i is characterized by the property that, for $\xi \in T_x^*M$ and $v \in E_x$, $i(\xi^k \otimes v) = j^k(\frac{1}{k!} \phi^k u)(x)$

if $\phi \in C^\infty(M)$ satisfies $\phi(x) = 0$ and $d\phi(x) = \xi$ and $u \in C^\infty(E)$ satisfies $u(x) = v$. Here ξ^k is the k^{th} symmetric tensor product. From now on we identify $i(S^k(T^*M) \otimes E)$ and $S^k(T^*M) \otimes E$. Introduce metrics in E and F and a Riemannian metric in M . We then choose a metric in $J^k(M, E)$ which on $S^k(T^*M) \otimes E$ is equal to the one induced from the metrics on M and E .

Now suppose that

$$\Phi : J^m(M, E) \longrightarrow F$$

is a smooth, fiber preserving map. Then

$$C^m(E) \ni u \longrightarrow \Phi(j^m u) \in C^0(F)$$

defines a (non-linear) differential operator, which, with a slight abuse of notations, we denote by $\Phi(u)$. (All differential operators in what follows will tacitly be assumed smooth.) Locally this means that if $e_1, \dots, e_N \in C^\infty(E)$, $v^* \in C^\infty(F^*)$, and $x = (x_1, \dots, x_n)$

is a system of local coordinates, then

$$\langle \Phi(\sum u_j e_j), v^* \rangle(x) = G(x, \{\partial^\alpha u_j(x); |\alpha| \leq m, 1 \leq j \leq N\}),$$

where G is smooth. We say that $\Phi(u)$ is a homogeneous polynomial of degree k if, for all choices of $e_1, \dots, e_N, v^*$ and local coordinates x , G is a homogeneous polynomial of degree k in $\{\partial^\alpha u_j(x); |\alpha| \leq m, 1 \leq j \leq N\}$.

Now suppose that Φ is a homogeneous polynomial of degree k and is defined for all $u \in C^m(E)$. Then, if $\phi \in C^\infty(M)$, $e^{-tk\phi(x)} \Phi(e^{t\phi} u)(x)$ is a polynomial in t of degree at most km . The coefficient of t^{km} only depends on $d\phi(x)$ and $u(x)$ and is a homogeneous polynomial of degree k in $u(x)$. This means that for every $\xi \in T_x^*M$ we can define as follows an element in $S^k(E_x^*) \otimes F_x$. Let $\phi \in C^\infty(M)$ with $d\phi(x) = \xi$, $u \in C^\infty(E)$ with $u(x) = v$, and form

$$\Phi_{km}(\xi)(v) = \lim_{t \rightarrow \infty} t^{-km} e^{-tk\phi(x)} \Phi(e^{t\phi} u)(x).$$

When ξ varies this defines a section, called the principal symbol of Φ ,

$$\Phi_{km}: T^*M \rightarrow \pi^*(S^k(E^*) \otimes F),$$

which is a homogeneous polynomial of degree km in the fiber of T^*M . Here $\pi: T^*M \rightarrow M$ is the projection from the cotangent bundle.

It is worth noting that the principal symbol can vanish identically without the operator being zero. For example, let M be the n -torus T^n , $n \geq 2$, and consider for $u \in C^1(T^n, \mathbb{R}^n)$ the operator $\Phi(u)(x) = \det(\partial_k u_j(x))$. Then $\Phi_{km}(\xi)(v) = \det(v_j \xi_k) = 0$.

Now let Φ be a general differential operator of order m defined in some open set $\mathcal{U}$ in $C^m(E)$ and suppose that u and v

are sections of E such that $u \in \mathcal{U}$. Then we have a formal power series expansion

$$\Phi(u+\epsilon v) \sim \sum_{k=0}^{\infty} \Phi^{(k)}(u;v) \epsilon^k/k!,$$

where $v \rightarrow \Phi^{(k)}(u;v)$ is a homogeneous polynomial of degree k , defined on $C^m(E)$. Every such operator has a principal symbol $\Phi_{km}^{(k)}(u)$. In this note we are mainly concerned with the first two symbols, that is, the linear principal symbol

$$\Phi_m'(u): T^*M \rightarrow \pi^*(E^* \otimes F)$$

and the quadratic symbol

$$\Phi_{2m}''(u): T^*M \rightarrow \pi^*(S^2(E^*) \otimes F).$$

We shall now define a certain conic set in F , depending on Φ , which will play an important part in Theorem 3.1. Let S^1 be the circle $\mathbb{R}/\mathbb{Z}$ and let $p: M \times S^1 \rightarrow M$ be the projection on the first factor. Suppose Φ is given.

Definition 2.1. Let $u \in C^\infty(E)$ and $\phi \in C^\infty(M)$ be given. Then $w \in C^\infty(p^*E)$ is called a local curve for u and ϕ if

$$(2.1) \quad \int_0^1 w(x,y) dy = 0 \quad \text{for every } x \text{ in } M,$$

$$(2.2) \quad \Phi_m'(u; d\phi)w = 0 \quad \text{in } M \times S^1,$$

$$(2.3) \quad \Phi_{2m}''(u; d\phi)(w) \quad \text{is a function in } M \text{ lifted to } M \times S^1.$$

By means of this concept we now define a set in the vector space F_x as follows.

Definition 2.2. Let $\dot{u}_0 \in C^m(E)$ be given. Then $\Gamma_x^0(u_0) \subset F_x$ is defined as the set of all $f \in F_x$ such that there exist

- (i) a neighborhood $\mathcal{U}$ of u_0 in C^m ,
- (ii) a constant K ,
- (iii) a non-linear differential operator $R(u)$, $u \in \mathcal{U}$, of order m , with $R(u_0)(x) = f$,
- (iv) for every $u \in \mathcal{U} \cap C^\infty$ a function $\phi \in C^\infty(M)$ and a local curve w for u and ϕ

satisfying the following conditions

$$(2.4) \quad \Phi_{2m}''(u; d\phi)(w) = R(u), \quad u \in \mathcal{U},$$

$$(2.5) \quad |w| \leq K, \quad |d\phi| \leq K.$$

Definition 2.3. The set $\Gamma(u)$ in F is defined as the interior of

$$\bigcup_{x \in M} \text{convex hull } \Gamma_x^0(u).$$

For the proof of Theorem 3.1 we need the following lemma, which is a direct consequence of definition 2.2.

Lemma 2.4. Suppose $u_0 \in C^m(E)$ and $f \in C^0(F)$ with image in $\Gamma(u_0)$ are given. Then there is a C^m neighborhood $\mathcal{U}_0$ of u_0 , $\delta > 0$, $K > 0$, differential operators $R_1(u), \dots, R_I(u)$, $u \in \mathcal{U}_0$, of order m and smooth functions A_i , $i = 1, \dots, I$, on $\frac{I}{0} F$ with $|A_i| \leq 1$, such that, for every $u \in \mathcal{U}_0 \cap C^\infty$ and $g \in C^0(F)$ with $\sup_M |g-f| < 2\delta$

- (i) the functions $a_i = A_i(., g, (R_k(u))_{k=1}^I)$ give a representation of g ,

$$(2.6) \quad g(x) = \sum_1^I a_i(x)^2 R_i(u)(x),$$

- (ii) there are $\phi_i \in C^\infty(M)$ and local curves w_i for u and ϕ_i such that $\Phi_{2m}''(u; d\phi_i)(w_i) = R_i(u)$ and

$$(2.7) \quad |w_i| \leq K, \quad |d\phi_i| \leq K, \quad i = 1, \dots, I.$$

Proof. Let $x_0 \in M$. Using a theorem of Carathéodory we can then find $f_i \in \Gamma_{x_0}^0(u_0)$, $i = 0, \dots, N'$, such that

$$f(x_0) = \sum_0^{N'} f_i / (N' + 1)$$

and $f_0, \dots, f_{N'}$, are affinely independent. According to definition 2.2 we can find a C^m neighborhood $\mathcal{U}$ of u_0 , a constant K and differential operators $R_i(u)$, $u \in \mathcal{U}$, $i = 0, \dots, N'$, of order m with $R_i(u_0)(x_0) = f_i$, such that for every $u \in \mathcal{U} \cap C^\infty$ there exist $\phi_i \in C^\infty(M)$ and local curves w_i for u and ϕ_i with $\phi_{2m}''(u; d\phi_i)(w_i) = R_i(u)$, $u \in \mathcal{U} \cap C^\infty$, and $|w_i| \leq K$, $|d\phi_i| \leq K$, $i = 0, \dots, N'$. From reasons of continuity it is clear that if we shrink $\mathcal{U}$ a bit, there is a neighborhood Ω of x_0 in M such that $R_i(u)(x)$, $i = 0, \dots, N'$, are affinely independent for $u \in \mathcal{U}$ and $x \in \Omega$. We can assume that $\mathcal{U}$ and Ω are chosen so small, that if $u \in \mathcal{U}$ and $x \in \Omega$ then $f(x)$ is in the interior of the convex hull of $R_i(u)(x)$, $i = 0, \dots, N'$.

We can therefore find $\delta_0 > 0$ such that if $g \in C^0(F)$ and $|(g-f)(x)| < 2\delta_0$ for x in Ω , then one can find $a_i(x)$ so that

$$\sum_0^{N'} a_i^2 = 1, \quad g(x) = \sum_0^{N'} a_i^2 R_i(u)(x), \quad x \in \Omega, \quad u \in \mathcal{U}.$$

This is a linear system for a_i^2 with a non-vanishing determinant, which shows that a_i is of the desired form locally. A standard covering argument using the Heine-Borel lemma and a partition of unity consisting of squares then gives the global statement.

We close this section with some examples.

Example 2.5. Let $u_0 \in C^\infty(E)$ and let $\mathcal{U}_0$ be a C^m neighborhood of u_0 such that

- (i) $\ker \Phi'_m(u; \xi)$ has dimension $\mu \geq 2$ for every $u \in \mathcal{U}_0$ and $\xi \in T^*M \setminus 0$,
- (ii) there are smooth fiber preserving maps

$$B: J^m(M, E) \oplus T^*M \rightarrow S^2(E^*), \quad R: J^m(M, E) \oplus T^*M \rightarrow F,$$

such that, with the usual abuse of notations, for every $u \in \mathcal{U}_0$ we have a factorisation

$$\Phi''_{2m}(u; \xi) = B(u; \xi) \boxtimes R(u; \xi)$$

on $\ker \Phi'_m(u; \xi)$. Here $R(u; \xi)$ is a homogeneous polynomial of degree $2m$ in the fibers of T^*M and $B(u; \xi)$ is positive definite for every $\xi \in T^*M \setminus 0$.

If $\xi \in T^*M \setminus 0$ and $v \in \ker \Phi'_m(u; \xi) \setminus 0$, then

$$\Phi''_{2m}(u; \xi)(v) = B(u; \xi)(v) R(u; \xi) = a R(u; \xi), \quad a > 0,$$

so it is clear that $\Gamma(u_0)$ is contained in the interior of

$\bigcup_{x \in M} \text{convex hull} \{R(u_0; \xi); \xi \in T^*_x M \setminus 0\}$. We shall show that there is in fact equality.

For this we first note that if we shrink $\mathcal{U}_0$ a bit, there is a constant K such that

$$(2.8) \quad |v|^2 \leq K B(u; \xi)(v), \quad v \in \ker \Phi'_m(u; \xi)$$

for every $u \in \mathcal{U}_0$ and $\xi \in T^*M \setminus 0$. This follows for continuity reasons, since $B(u; \xi)$ is homogeneous of degree 0 in the fibers of T^*M .

Now let us compute $\Gamma^0_{x_0}(u_0)$ for $x_0 \in M$. Let $u \in \mathcal{U}_0 \cap C^\infty$ and $\phi \in C^\infty$ with $d\phi(x_0) \neq 0$ be given. Then we can define a local curve for u and ϕ as follows. Let Ω be a simply connected

open neighborhood of x_0 and let ζ and η be smooth sections of $\ker \Phi'_m(u; d\phi)$ over Ω such that

$$(2.9) \quad B(u; d\phi)(\zeta, \zeta) = B(u; d\phi)(\eta, \eta) = 1, \quad B(u; d\phi)(\zeta, \eta) = 0.$$

Then define

$$w(x, y) = \rho(x)((\sin 2\pi y)\zeta(x) + (\cos 2\pi y)\eta(x)),$$

where $\rho \in C_0^\infty(\Omega)$, $0 \leq \rho \leq 1$ and $\rho(x_0) = 1$. Then (2.1) and (2.2) are immediate and since $\sin^2 + \cos^2 = 1$ we get that $\Phi''_{2m}(u; d\phi)(w) = \rho^2 R(u; d\phi)$ which proves (2.4). Since $|w| \leq 2 K^{1/2}$ according to (2.8) and (2.9) and since we use the same ϕ for different $u \in \mathcal{U}_0$, we also have (2.5) fulfilled. From this it now follows that $\Gamma_{x_0}^0(u_0)$ contains $\{R(u; \xi); \xi \in T_x^*M \setminus \{0\}\}$ and therefore equals it.

A special case of this is the embedding operator, where we have $B(u; \xi)(v) = |v|^2$ and $R(u; \xi) = \xi^2$. We will return to this in section 4.

Example 2.5 is close to the original ideas of Nash, since the local curves are circles $y \rightarrow (\sin 2\pi y)\zeta + (\cos 2\pi y)\eta$. In our other example, these periodic curves are not always sufficient, which justifies our more abstract version.

Example 2.6. If $N \geq 2(N'+1)$ we can find an operator for which there is a function $\phi \in C^\infty(M)$ such that $\Phi'_m(0; d\phi)$ has rank N' and such that $\Gamma(0) \cap F_x$ is non-empty for every x in M . (0 denotes the section $u = 0$.)

The following lemma, which we prove in section 5, is the key to the proof that there are such operators.

Lemma 2.7. Suppose $v \geq \mu + 2$. Let

$$Q_i(w, w) = \sum_{j \leq k} a_{ijk} w_j w_k, \quad i = 1, \dots, \mu,$$

be quadratic forms on $\mathbb{R}^v$ and let $a = \{a_{ijk}; 1 \leq i \leq \mu, 1 \leq j \leq k \leq v\}$. Then we can find a^0 such that for every a in a neighborhood of a^0 the set

$$X_a: Q_i(w, w) = 1, \quad i = 1, \dots, \mu,$$

contains a closed curve $w(t, a)$ with period 1, which depends smoothly on a , such that

$$\int_0^1 w(t, a) dt = 0,$$

Let us note that $t \rightarrow w(t, a)$ can not be a circle for every a if $2v \leq 3\mu$. To see this, note that the circle $(\sin t)\zeta + (\cos t)\eta$ lies in X_a if and only if

$$Q_i(\zeta, \zeta) = Q_i(\eta, \eta) = 1, \quad Q_i(\zeta, \eta) = 0, \quad i = 1, \dots, \mu.$$

This is 3μ equations for the coefficients a in $Q_1, \dots, Q_\mu$. Moreover, the dimension of the set of circles is $2v - 1$, since a given circle defines (ζ, η) apart from a rotation. If $2v - 1 < 3\mu$ it therefore follows from the Morse-Sard theorem that we can find a such that X_a does not contain a circle.

Once we have got this lemma, we can construct an operator

$$\Phi: C^m(\mathbb{T}^n, \mathbb{R}^N) \longrightarrow C^0(\mathbb{T}^n, \mathbb{R}^{N'}),$$

where $\mathbb{T}^n$ is the n -torus, in the following way. Let $\xi_0 \in \mathbb{R}^n$ and let $P(u)$ be a linear differential operator of order m with constant coefficients such that $p_m(\xi_0): \mathbb{R}^N \rightarrow \mathbb{R}^{N'}$ has maximal rank N' , where p_m is the principal symbol of P . Then let $\Psi(u)$ be a differential operator of order m which is a homogeneous polynomial of degree 2 with constant coefficients, such that $\Psi''_{2m}(\xi_0)$ on $\ker p_m(\xi_0) \subset \mathbb{R}^N$ is of the form $(Q_1, \dots, Q_\mu)$ in Lemma 2.7 with $v = N - N'$ and $\mu = N'$. Now consider, for u in a C^m neighborhood of 0, the operator

$$\Phi(u) = P(u) + \Psi(u).$$

Then $\Phi'_m(u; \xi_0) = p_m(\xi_0) + \Psi'_m(u; \xi_0)$ has maximal rank N' if u is sufficiently close to 0 in C^m , since $\Psi'_m(0; \xi_0) = 0$, and its kernel is close to $\ker p_m(\xi_0)$. Then we can use Lemma 2.7 on the restriction of $\Psi''_{2m}(\xi_0)$ to $\ker \Phi'_m(u; \xi_0)$, viewed as a system on $\mathbb{R}^{N-N'}$, to get the desired result. If $2(N'+1) \leq N \leq 5N'/2$ we obtain examples not included in Example 2.5.

3. The implicit function theorem. With the notations in section 2 we now want to prove

Theorem 3.1. Let $u_0 \in C^\infty(E)$ and let $f \in C^0(F)$ have its image in $\Gamma(u_0)$. Then we can find $t > 0$ so that the equation

$$\phi(u) = \phi(u_0) + tf$$

has a C^m solution u .

Remark. The proof actually shows that there is a C^0 neighborhood $\mathcal{V}$ of f and $t > 0$ such that for every $f' \in \mathcal{V}$ and $t' \leq t$ the equation $\phi(u) = \phi(u_0) + t'f'$ has a C^m solution u .

It should also be noted that the restriction to compact manifolds is not essential, only introduced to shorten the proof.

Proof. Let $f \in C^0(F)$ be a section whose image is in $\Gamma(u_0)$ and put $f_0 = \phi(u_0)$. Use Lemma 2.4 to fix a C^m neighborhood $\mathcal{U}_0$ of u_0 and $\delta > 0$. Then we want to find sections $u_k \in \mathcal{U}_0 \cap C^\infty(E)$ such that $\phi(u_k)$ approximates $f_0 + t \sum_{j=1}^k f_j$, where we have put $f_j = 2^{-j}f$ when $j > 0$. The error

$$e_k = \phi(u_k) - (f_0 + t \sum_{j=1}^k f_j),$$

is to be much smaller than tf_{k+1} . It suffices to construct sections u_k so that they fulfill the conditions in the following lemma.

Lemma 3.2. We can find sections $u_v \in \mathcal{U}_0 \cap C^\infty(E)$, $v = 1, 2, \dots$ and positive constants t and C such that for every x in M

$$(3.1) \quad |e_v(x)| \leq t\delta 2^{-(v+1)},$$

$$(3.2) \quad |j^m(u_v - u_{v-1})(x)| \leq C \sqrt{\epsilon} 2^{-v/2}.$$

This proves Theorem 3.2 if we simply set $u = u_0 + \sum_{j=1}^{\infty} (u_j - u_{j-1}) = \lim u_v$, since the sum converges in C^m by (3.2).

Proof. Suppose $u_0, \dots, u_k \in \mathcal{L}_0 \cap C^\infty$ have been constructed. We then want to construct u_{k+1} in $\mathcal{L}_0 \cap C^\infty$. For this we first want to approximate $tf_{k+1} - e_k$ with a smooth section g . We therefore note that

$$e_{k+1} = \phi(u_{k+1}) - \phi(u_k) - g + (g - (tf_{k+1} - e_k))$$

and that it follows from (3.1) that $|f - (f - 2^{k+1}e_k/t)| < 2\delta$, so we can find $g \in C^\infty(F)$ such that $|f - 2^{k+1}g/t| < 2\delta$ and

$$(3.3) \quad |tf_{k+1} - e_k - g| \leq \frac{1}{2}t\delta 2^{-(k+2)}.$$

It then remains to find u_{k+1} such that (3.2) is valid for $v = k + 1$ and

$$(3.4) \quad |\phi(u_{k+1}) - \phi(u_k) - g| \leq \frac{1}{2}t\delta 2^{-(k+2)}.$$

According to the definition of g and (2.6) we can write with $a_i \in C^\infty$

$$(3.5) \quad g = \sum_1^I a_i^2 R_i(u_k), \quad |a_i| \leq \sqrt{t} 2^{-(k+1)/2}, \quad i = 1, \dots, I.$$

We shall now define $u'_0, \dots, u'_I \in \mathcal{L}_0 \cap C^\infty$ so that $u'_0 = u_k$ and so that $u_{k+1} = u'_I$ fulfills Lemma 3.2. Now suppose that $u'_0, \dots, u'_{i-1}$ have been constructed and that we want to construct u'_i . Because of Lemma 2.4 we can find ϕ_i and w_i such that (replacing the w_i of Lemma 2.4 by $\sqrt{2} w_i$)

$$(3.6) \quad \frac{1}{2} \phi''_{2m}(u'_{i,1}; d\phi_i)(w_i) = R_i(u'_{i,1})$$

Now let

$$W_i(x, y) = \int_0^1 w_i(x, y-z) B_m(z)/m! dz,$$

where B_m is the m^{th} Bernoulli polynomial, be the m^{th} integral of w_i with respect to y with mean value zero. Define

$$u_i^! = u_{i-1}^! + v_i(\theta), \text{ where}$$

$$v_i(\theta) = a_i W_i(., \theta \phi_i) / \theta^m.$$

Then we have

$$j^m v_i(\theta) - a_i w_i(., \theta \phi_i) (d\phi_i)^m = O(\theta^{-1}),$$

so according to (2.7) and the slightly modified definition of w_i we then get

$$(3.7) \quad |j^m v_i(\theta)| \leq (2K)^{m+1} \sqrt{t} 2^{-(k+1)/2}$$

if we take θ sufficiently large. Since $u_{k+1} = u_k + \sum_1^I v_i$, this shows that we can get (3.2) with $C = I(2K)^{m+1}$ if we take θ sufficiently large in the different steps.

It remains only to show that, if θ is chosen large enough and t is sufficiently small, we can also get (3.4). For this we consider the function

$$h(s) = \Phi(u_{i-1}^! + s v_i(\theta)) - \Phi(u_{i-1}^!) - s \Phi'(u_{i-1}^!) v_i(\theta).$$

Then

$$h(1) = \int_0^1 (1-s) h''(s) ds,$$

that is, with the notation $w_i(\theta) = w_i(., \theta \phi_i)$,

$$\begin{aligned} & \Phi(u_i^!) - \Phi(u_{i-1}^!) - \frac{1}{2} a_i^2 \Phi_{2m}''(u_{i-1}^!; d\phi_i)(w_i(\theta)) = \Phi'(u_{i-1}^!) v_i(\theta) + \\ & + \int_0^1 (1-s) (\Phi''(u_{i-1}^! + s v_i(\theta))(v_i(\theta)) - a_i^2 \Phi_{2m}''(u_{i-1}^! + s v(\theta); d\phi_i)(w_i(\theta))) ds \\ & + a_i^2 \int_0^1 (1-s) (\Phi_{2m}''(u_{i-1}^! + s v_i(\theta); d\phi_i)(w_i(\theta)) - \Phi_{2m}''(u_{i-1}^!; d\phi_i)(w_i(\theta))) ds. \end{aligned}$$

Here the first two terms on the right hand side clearly are $O(\theta^{-1})$ because of (2.2) and the definition of principal symbols. By taking θ large we can therefore get them bounded by $t\delta 2^{-(k+2)}/8I$. For the third term, we note that the map

$$u \longrightarrow \Phi_{2m}''(u; d\phi_i)(w_i(\theta))$$

has a differential which is uniformly bounded in $\mathcal{U}_0$, since $|w_i(\theta)| \leq K$. Therefore we can find a constant C_1 such that, for $0 \leq s \leq 1$,

$$\begin{aligned} & |a_i^2(\Phi_{2m}''(u_{i-1}^! + sv_i(\theta); d\phi_i)(w_i(\theta)) - \Phi_{2m}''(u_{i-1}^!; d\phi_i)(w_i(\theta)))| \leq \\ & \leq C_1 |a_i|^2 |j^m v_i(\theta)| \leq C_2 (t 2^{-(k+1)})^{3/2}. \end{aligned}$$

Here we used (3.7) and (3.5) for the last inequality. By choosing t small, we can get this bounded by $t\delta 2^{-(k+2)}/4I$. This shows that

$$(3.8) \quad |\Phi(u_i^!) - \Phi(u_{i-1}^!) - \frac{1}{2} a_i^2 \Phi_{2m}''(u_{i-1}^!; d\phi_i)(w_i(\theta))| \leq t\delta 2^{-(k+2)}/4I$$

if θ is sufficiently large and t sufficiently small. Now recall that $\frac{1}{2} \Phi_{2m}''(u_{i-1}^!; d\phi_i)(w_i(\theta)) = R_i(u_{i-1}^!)$ for every θ . Since

R_i is a differential operator of order m , we have an estimate

$$|R_i(u_{i-1}^!) - R_i(u_0^!)| \leq C_3 |j^m(u_{i-1}^! - u_0^!)| \leq C_4 \sqrt{t} 2^{-(k+1)/2},$$

so that if t is sufficiently small,

$$(3.9) \quad |a_i^2(R_i(u_{i-1}^!) - R_i(u_0^!))| \leq t\delta 2^{-(k+2)}/4I.$$

Summing over i and using (3.5), (3.8) and (3.9) we get (3.4).

Finally we note that $|j^m(u_{k+1} - u_0)| \leq C \sqrt{t}$, so if t is chosen sufficiently small, we can make sure that $u_{k+1} \in \mathcal{U}_0$. This completes the proof of Lemma 3.2.

Remark. Note that if $\phi_{2m}''(u)$ is independent of u , we do not need to take t small. In fact, in Lemma 2.4 we do not need u_0 then, and the terms estimated at the end of the proof of Theorem 3.1 in which we took t small, are identically zero. This shows that in that case the equation $\phi(u) = \phi(u_0) + f$ always has a solution u in C^m if f takes its values in $\Gamma = \Gamma(u_0)$. This we will use in the next section.

4. The embedding operator. Let M be a compact C^∞ manifold of dimension n and consider for $u \in C^1(M, \mathbb{R}^N)$, $N \geq n+2$, the operator

$$\Phi(u) = \langle du, du \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^N$. This can be considered as an operator from $C^1(M \times \mathbb{R}^N)$ to $C^0(S^2(T^*M))$, where $M \times \mathbb{R}^N$ is the trivial N -dimensional bundle over M .

Then $\Phi'(u)v = 2\langle du, dv \rangle$, which means that $\Phi'_m(u; \xi)r = 2\langle du, \xi \otimes r \rangle$, or explicitly, that if $t_1, t_2 \in T_x M$ and $r \in \mathbb{R}^N$ then

$$\Phi'_m(u; \xi)r(t_1, t_2) = \langle du(t_1), r \rangle \xi(t_2) + \langle du(t_2), r \rangle \xi(t_1).$$

If $v \in \ker \Phi'_m(u; \xi)$ then $\xi(t)\langle du(t), r \rangle = 0$ so if $\xi \neq 0$ we obtain

$$\langle du(t), r \rangle = 0 \text{ for all } t,$$

that is, r is normal to $u(M)$ in $\mathbb{R}^N$. Conversely this implies $r \in \ker \Phi'_m(u; \xi)$ so $\ker \Phi'_m(u; \xi)$ has dimension $N-n \geq 2$ if u is a C^1 immersion. Note that C^1 immersions form an open set in $C^1(M, \mathbb{R}^N)$.

We next consider the quadratic part. Since $\Phi''(u)(v) = \langle dv, dv \rangle$ we get

$$\Phi''_{2m}(u; \xi)(r_1, r_2) = \langle r_1, r_2 \rangle \xi^2, \quad r_1, r_2 \in \mathbb{R}^N,$$

that is, in the notations of example 2.5,

$$B(u; \xi) = \text{identity of } S^2(\mathbb{R}^N), \quad R(u; \xi) = \xi^2.$$

Comparing with example 2.5 we thus see that $\Gamma(u) \cap S^2(T_x^*M)$, for u a C^1 immersion, is the interior of the convex hull of $\{\xi^2; \xi \in T_x^*M \setminus \{0\}\}$, that is, the set of positive definite forms in $S^2(T_x^*M)$. Furthermore, $R(u; \xi)$ does not depend on u , so Theorem 3.1 and the remark after the proof give

Theorem 4.1. (Nash) Let $g \in C^0(S^2(T^*M))$ be a given positive definite metric. If there is an immersion $u_0 \in C^\infty(M, \mathbb{R}^N)$, $N \geq n+2$, then there is $u \in C^1(M, \mathbb{R}^N)$ such that $\langle du, du \rangle = g$.

The only thing to note in order to complete the proof is that if $u_0 \in C^\infty$ is an immersion in $\mathbb{R}^N$, and $s > 0$ is sufficiently small, then $g - s^2 \langle du_0, du_0 \rangle$ is positive definite, that is, belongs to $\Gamma(su_0)$.

Nash actually proved more, namely that we can substitute embedding for immersion in Theorem 4.1. This can rather easily be deduced from the proof of Theorem 3.1; to get the injectivity one uses that sections are altered in normal directions in the iteration process. He also proved Theorem 4.1 for non-compact manifolds. It should also be noted that the lower bound $n+2$ for N can be reduced to $n+1$, as proved by Kuiper [2].

5. Proof of Lemma 2.7. In the proof we denote the variable in $\mathbb{R}^v$ by x ; the proof then consists of the following sequence of lemmas.

Lemma 5.1. If $\mu \leq v - 2$, we can find linear forms $L_1, \dots, L_\mu$ on $\mathbb{R}^v$ such that the set

$$X_0: L_i(x_1^2, \dots, x_v^2) = 1, \quad i = 1, \dots, \mu,$$

contains in its regular part a closed smooth curve whose convex hull has the origin as an interior point.

Proof. It suffices to prove the lemma for the case $v = \mu + 2$. For this, let $L_i^!(z_1, z_2) = a_i z_1 + b_i z_2$, $i = 1, \dots, \mu$, be linear forms on $\mathbb{R}^2$ such that the set

$$K = \{z \in \mathbb{R}^2; z_k \geq 0, k = 1, 2, \text{ and } L_i^!(z) \leq 1, i = 1, \dots, \mu\}$$

is a $\mu + 2$ -sided bounded convex set. Then define the forms

$$L_i(y) = y_i + L_i^!(y_{\mu+1}, y_{\mu+2}), \quad i = 1, \dots, \mu,$$

on $\mathbb{R}^{\mu+2}$, and consider the set

$$X_0: L_i(x_1^2, \dots, x_{\mu+2}^2) = 1, \quad i = 1, \dots, \mu.$$

The singular part $\Sigma(X_0)$ of X_0 is the set of points where the matrix

$$\begin{pmatrix} x_1 & 0 & \dots & 0 & a_1 x_{\mu+1} & b_1 x_{\mu+2} \\ 0 & x_2 & \dots & 0 & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & & x_\mu & a_\mu x_{\mu+1} & b_\mu x_{\mu+2} \end{pmatrix}$$

does not have rank μ . But this means that at least two x -variables vanish. In fact, the product $x_1 \dots x_\mu$ vanishes and if, say, only $x_1 = 0$, then $a_1 x_{\mu+1} = b_1 x_{\mu+2} = 0$ for otherwise we would have μ linearly independent rows. The image of $\Sigma(X_0)$ under the map

$$(5.1) \quad X_0 \ni (x_1, \dots, x_{\mu+2}) \longrightarrow (x_{\mu+1}^2, x_{\mu+2}^2) \in K$$

is therefore contained in the corners of K .

We now construct a curve γ on X_0 in the following way. Take a point in the interior of K and draw a smooth curve to the edge $z_1 = 0$ (not one of the corners). This lifts to X_0 by (5.1) to a smooth curve which can be smoothly continued through the hyperplane $x_1 = 0$. Prolong the image by the map (5.1) of this curve to the side $z_2 = 0$ and repeat the procedure for the sides $L_i^+(z) = 1$ also. In this way we can draw a curve between the edges of K , avoiding the corners, so that the lift by the procedure above is a curve in X_0 that enters all the $2^{\mu+2}$ "octants" and lies completely in the regular part of X_0 . This proves Lemma 5.1.

Lemma 5.2. Let γ be the image of a closed, smooth curve in $\mathbb{R}^v$ such that the origin is an interior point of the convex hull of γ . For $\epsilon \in \mathbb{R}^v$ in a neighborhood of 0 we can then find curves $x(t, \epsilon)$ with period 1 in $t \in \mathbb{R}$, which depend smoothly on ϵ , such that $x([0, 1], \epsilon) = \gamma$ for every ϵ and

$$\int_0^1 x(t, \epsilon) dt = \epsilon.$$

Proof. Let $x(t)$ be a parametrisation of γ such that $x(t)$ has period 1. According to a theorem of Carathéodory we can then find $v+1$ points $t_0, \dots, t_v$ such that $x(t_j)$, $j = 0, \dots, v$, are affinely independent and $\sum_{j=0}^v c_j x(t_j) = 0$ for some positive c_j

with $\sum c_j = 1$. If $\phi_j \in C_0^\infty(\mathbb{R})$, $\phi_j \geq 0$, $\int \phi_j(t) dt = 1$ and ϕ_j is a sufficiently good approximation of the Dirac measure $\delta(t_j)$, then 0 is also an interior point of the convex hull of the points

$$A_j = \int x(t) \phi_j(t) dt, \quad j = 0, \dots, v.$$

Let B be the ball $|x| < r$ in $\mathbb{R}^v$ with r chosen so small that the ball $|x| < 3r$ is contained in the convex hull of the points $A_0, \dots, A_v$. Let $A = \int_0^1 x(t) dt$. Take $0 < \delta < 1$ so small that δA belongs to B , and define, for $c = (c_0, \dots, c_v)$, $0 \leq c_j \leq 1$, $\sum c_j = 1$, the map S_c from $[0, 1]$ to itself by

$$S_c'(t) = (\delta + \sum_0^v c_j \phi_j(t)) / (1 + \delta).$$

Let $x_c(s)$ denote the curve defined by $x_c(S_c(t)) = x(t)$. Then we want to find $c = c(\epsilon)$ so that $\int_0^1 x_c(s) ds = \epsilon$. This means that we want to solve the linear system of equations

$$\sum_0^v c_j A_j = (1 + \delta)\epsilon - \delta A, \quad \sum_0^v c_j = 1,$$

which, by our assumptions, can be done for ϵ in B . This proves Lemma 5.2.

Lemma 5.3. Let $L_1, \dots, L_\mu$ be the linear forms of Lemma 5.1 and let

$$Q_i(x, x) = \sum_{j < k} a_{ijk} x_j x_k, \quad i = 1, \dots, \mu,$$

be general quadratic forms on $\mathbb{R}^v$. Let

$$X_a: L_i(x_1^2, \dots, x_v^2) + Q_i(x, x) = 1, \quad i = 1, \dots, \mu,$$

where $a = \{a_{ijk}; 1 \leq i \leq \mu, 1 \leq j < k \leq v\}$. Then we can find curves $x(t, a)$ on X_a with period 1, depending smoothly on a for a in a neighborhood of 0, such that

$$\int_0^1 x(t,a)dt = 0.$$

Proof. Let γ be the image of a closed, smooth curve in the regular part X_{reg} of X_0 , such that the origin is an interior point of the convex hull of γ . Let K be a compact neighborhood of γ in X_{reg} and let N be the normal bundle of X_{reg} . We can then find smooth sections n_a of N over K with $n_0 = 0$, which depend smoothly on a for a in a neighborhood of 0 and are such that $x + n_a(x)$ lies in X_a for all x in K . This follows from the implicit function theorem.

Now, for $\varepsilon \in \mathbb{R}^V$ in a neighborhood of 0 , let $x(t,\varepsilon)$ be smooth closed curves with image γ , period 1 and $\int_0^1 x(t,\varepsilon)dt = \varepsilon$. These exist by Lemma 5.2. We then define $x(t,a,\varepsilon) = x(t,\varepsilon) + n_a(x(t,\varepsilon))$ and note that for the function

$$F(a,\varepsilon) = \int_0^1 x(t,a,\varepsilon)dt = \varepsilon + \int_0^1 n_a(x(t,\varepsilon))dt$$

we have $F'_\varepsilon(0,0) = \text{identity}$, since $F(0,\varepsilon) = \varepsilon$. By the implicit function theorem we can therefore define $\varepsilon = \varepsilon(a)$ so that $F(a,\varepsilon(a)) = 0$ for a in a neighborhood of zero. This proves Lemma 5.3.

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